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Design of Mode Transducers*

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Summary—**The propagation of the electromagnetic wave in a gradual transducer is discussed. It is shown that the incident mode and the geometry of the transducer determine the outgoing mode. Inverting this theorem, a method is suggested for the design of the transducer's surface for cases in which the desired modes in the uniform waveguides are given.**

The application of the method is illustrated in three examples.

I. INTRODUCTION

IN the design of a microwave transmission system it is often necessary to connect two uniform waveguides of different cross section by means of a nonuniform waveguide (subsequently referred to as a transducer). The transducer can be used for two different purposes: 1) to transform the same mode from one waveguide into another waveguide of different size; and 2) to transform a certain mode of one waveguide into a predetermined mode of the other waveguide.

The best example for the first type is a transducer between two rectangular waveguides of different size. The requirement is to transform efficiently the H_{01} mode in a specified bandwidth. All the solutions naturally employ a transducer whose cross section is everywhere rectangular. Similarly, the cross section of a transducer between two circular waveguides of different diameter is always circular. The problem in these cases is how to vary the size of the cross section. This field is well explored, and for certain cases optimum solutions have been obtained.

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The design of a transducer of the second type (generally called a mode transducer) is incomparably more complicated, since the shape of the cross section is varying. Although mode transducers have been used since the earliest days of microwave transmission, no systematic procedure seems to have been developed for the design of the required cross sections. The existing mode transducers were designed by physical intuition.

The aim of the present paper is to suggest a systematic design method. For the better understanding of the basic phenomena, the properties of a given transducer are first analyzed. It is shown that the incident mode and the surface of a sufficiently gradual transducer determine the outgoing mode. In the third section the inverse problem is dealt with, *i.e.*, choosing the surface of the transducer when the desired modes in the uniform waveguides are given.

II. THE PROPAGATION OF THE ELECTROMAGNETIC WAVE IN A SUFFICIENTLY GRADUAL TRANSDUCER

Let us consider the following arrangement of waveguides (see Fig. 1). The uniform waveguide A extends from $z=-\infty$ to $z=0$, the transducer from $z=0$ to $z=L$ and the uniform waveguide B from $z=L$ to $z=\infty$.

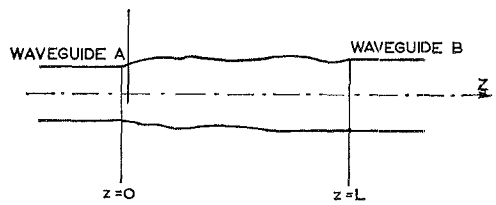

Fig. 1.

The transducer has the following properties:

a) The equation of the surface is differentiable as a function of z.

b) A plane perpendicular to the z axis cuts the surface in a single closed curve.

c) The cross sections [denoted by $S(z)$] at $z=0$ and $z=L$ are equal to those of the uniform waveguides A and B, respectively; *i.e.*, $S(O)=S^A$ and $S(L)=S^B$.

The propagation of an electromagnetic wave in this waveguide system can be studied with the aid of the equivalent circuit concept.

As is known,[1] a uniform waveguide can be represented by a set of uniform transmission lines, where each transmission line corresponds to a mode. The impedance and the propagation coefficient of the transmission line can be expressed by the eigenfunctions (cross-sectional wave functions) and eigenvalues (cutoff wave numbers) of the uniform waveguide. A wave can separately propagate on any of the transmission lines.

Propagation in a waveguide of gradually varying cross section can also be represented by the same picture of a number of transmission lines,[2,3] but now each line has gradually varying characteristics and there is a coupling between the lines. The coupling coefficients and the characteristics of the transmission lines can be expressed by the eigenfunctions and eigenvalues of a uniform waveguide, whose cross section is identical with that of the nonuniform waveguide at z. However, if the transducer is sufficiently gradual, the coupling between the transmission lines can be neglected.[4]

Thus, if a single mode enters the transducer from waveguide A it will travel along one of the transmission lines and emerge at the end as a single mode of waveguide B. However, at this stage of the argument it is not at all clear which mode of waveguide B will be excited; in order to discover this, the transducer has to be studied step by step. This can be done in principle (and in practice numerically) by determining the eigenfunctions and eigenvalues at a number of successive cross sections of the transducer; thus, the continuity of the transmission lines is determined in particular that of the transmission line which is terminated in the incident mode. The investigated cross sections must be spaced sufficiently close, so that the continuity can be clearly established.

If we write $t_m{}^A$ for the incident mode and refer to the transmission line on which it travels as the mth transmission line, and if $\psi_m(x, y, z)$ is the corresponding gradually varying eigenfunction, then the requirements on ψ_m are as follows:

1) It is differentiable as a function of z.
2) $\psi_m(x, y, 0)=\psi_m{}^A$, where $\psi_m{}^A$ is the eigenfunction of the $t_m{}^A$ mode in waveguide A.
3) $\psi_m(x, y, L)=\psi_m{}^B$, where $\psi_m{}^B$ is the eigenfunction of the excited mode in waveguide B (denoted by $t_m{}^B$).
4) It satisfies the two-dimensional wave equation for all values of z.
5) It satisfies the boundary conditions on the boundary of every cross section.

We must note here that as a direct consequence of the equivalent circuit of the transducer, the $t_m{}^B$ mode belongs to the same family (H or E) as the $t_m{}^A$ mode.

Summarizing the conclusions of this section, we state that if the geometry of a sufficiently gradual transducer and the incident mode are given, the outgoing mode may be determined with the aid of the $\psi_m(x, y, z)$ function.

However, the above treatment is not completely general. In a few special cases, spurious modes will be present in waveguide B even for a very gradual transducer. These exceptions are discussed in the Appendix.

III. Outline of the Design Method

We now invert the problem; instead of starting with an existing transducer and deducing $\psi_m(x, y, z)$, we lay down the required $t_m{}^A$ and $t_m{}^B$ modes, construct the eigenfunction of the mth transmission line, and design the transducer.

Knowing $\psi_m{}^A$ and $\psi_m{}^B$ it is always possible to construct a function ψ_a which satisfies conditions 1)–4). (A particular method of this construction will be given in a later section.) ψ_a becomes the eigenfunction of the mth transmission line, if it satisfies condition 5) as well as conditions 1)–4). Thus, the surface of the transducer must be designed in such a way that the boundary conditions for ψ_a are satisfied at every cross section. If the transducer furthermore satisfies conditions a)–c), then according to the analysis of the previous section, the transducer transforms the $t_m{}^A$ mode of waveguide A into the desired $t_m{}^B$ mode of waveguide B.

The Equation of Possible Boundaries

If both $t_m{}^A$ and $t_m{}^B$ belong to the family of the E modes, then the equation of the possible boundaries is given simply as follows:

$$\psi_m(x, y, z) = 0. \tag{1}$$

If both $t_m{}^A$ and $t_m{}^B$ are H modes, then the normal derivative of the eigenfunction should vanish at the boundary. Denoting by $f(x, y)=0$ the equation of the boundary curve at a given z, it must satisfy the following partial differential equation:

$$\frac{\partial \psi_m}{\partial x}\frac{\partial f}{\partial x}+\frac{\partial \psi_m}{\partial y}\frac{\partial f}{\partial y} = 0. \tag{2}$$

[1] N. Marcuvitz, "Waveguide Handbook," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 3–7; 1948.

[2] S. A. Schelkunoff, "Conversion of Maxwell's equations into generalised telegraphist's equations," *Bell Sys. Tech. J.*, vol. 34, pp. 995–1045; September, 1955.

[3] G. Reiter, "Connection of Two Waveguides by a Waveguide of Variable Cross-Section," M.S. thesis in applied mathematics, University Eotvos Lorand, Budapest, Hungary; June, 1955.

[4] L. Solymar, "Spurious mode generation in nonuniform waveguide," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 379–383; July, 1959.

In a practical case it is generally sufficient to construct the boundary curve by finding graphically the orthogonal trajectories of the (ψ_m=constant) electric lines.

Construction of $\psi_m(x, y, z)$

The eigenfunction of the mth transmission line can be constructed in an infinite number of ways. We shall choose it for most of the subsequent examples in a simple mathematical form as follows:

$$\psi_m(x, y, z) = g_1(z)\psi_m{}^A + g_2(z)\psi_m{}^B \quad (3)$$

where

$$g_1(0) = 1, \quad g_1(L) = 0$$
$$g_2(0) = 0, \quad g_2(L) = 1 \quad (4)$$

and both $g_1(z)$ and $g_2(z)$ are monotonic differentiable functions. ψ_m—constructed in this way—obviously satisfies conditions 1)–3). A simple (but not the only possible) way of meeting condition 4) is to make the eigenvalues (cutoff wave numbers) of the $t_m{}^A$ and $t_m{}^B$ modes equal. This implies a certain relation between the dimensions of waveguides A and B which may not be convenient. In practical devices, this can be overcome by a preliminary taper in which the dimensions of waveguide A (or B) are changed gradually, where the shape is kept the same.

An interesting special case arises when the eigenfunction of the $t_m{}^A$ and $t_m{}^B$ modes is the same, although the cross sections of the uniform waveguides are different. Then an obvious choice for the eigenfunction of the mth transmission line is

$$\psi_m = \psi_m{}^A = \psi_m{}^B. \quad (5)$$

Mode Purity

A disadvantage of the method is that we cannot perform the design for a specified mode purity. Because of the coupling (neglected in the design) between the transmission lines, spurious modes will always be present. After the transducer has been constructed, the power in the spurious modes is calculable[3] although the calculations are very laborious.

The mode purity will also depend on frequency, but since the transducer is built up by gradual change the purity of the desired mode cannot change violently with frequency. However, when the frequency is increased, the power in the spurious modes generally increases[4,5] due to the decrease in the difference of the propagation coefficients. Nevertheless, this increase in the power of the spurious modes is small within the normally required bandwidth and is unlikely to affect the performance.

[5] B. Z. Katzenelenbaum, "On the Theory of Nonuniform Waveguides with Slowly Changing Parameters," presented at the Congres International Circuits et Antennes Hyperfrequences, Paris, France; October, 1957.

Examples

We shall illustrate the design method in three examples.

The first example is a transducer between two uniform waveguides having cross sections which are, respectively, a section of a circle and a whole circle. The electric lines of the desired modes (both are H modes) are shown in Figs. 2(a) and 2(c). (The density of the lines

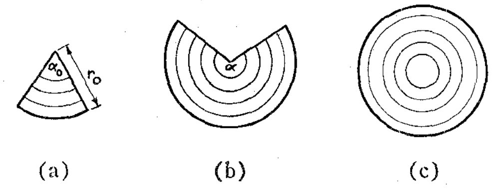

Fig. 2.

in these and subsequent drawings is not related to the intensity of the electric field). In this case, the eigenfunctions of both modes are the same. The simplest choice for the eigenfunction of the mth transmission line is in the form of (5). Then

$$\psi_m = J_0\left(3.83 \frac{\rho}{r_0}\right) \quad (6)$$

where

J_0 = zero order Bessel function,
ρ = the radial polar coordinate, and
r_0 = the radius of the circle.

The boundary of an intermediate cross section must be chosen in such a way that the boundary conditions are fulfilled for ψ_m. It may be easily shown that the normal derivative of (5) vanishes on the boundary of any section of a circle of radius r_0. Hence, the cross section of the transducer is chosen as a section of a circle of radius r_0, whose central angle is a monotonic function of z satisfying the conditions $\alpha(0) = \alpha_0$ and $\alpha(L) = 2\pi$.

This type of transducer is in common use.[6] The example simply shows that, using this method, we arrive at the same transducer.

We wish, however, to emphasize the fact that this is not the only solution. If we choose a different ψ_m we arrive at a different transducer. Another possible choice of ψ_m [it obviously satisfies conditions 1)–4)] is, for example,

$$\psi_m = J_0\left(3.83 \frac{\rho}{r(z)}\right) \quad (7)$$

where $r(z)$ is a slowly varying function of z, $r(0) = r(L) = r_0$. An intermediate cross section of this transducer is a section of a circle of radius $r(z)$.

[6] A. C. Beck, "Measurement Techniques for Multimode Waveguides," presented at the Symposium on Modern Advances in Microwave Techniques, New York, N. Y.; November, 1954.

A further choice of ψ_m might result in a transducer in which none of the intermediate cross sections is a section of a circle, and which nevertheless produces an arbitrarily pure mode at the output, provided that the transducer is sufficiently gradual.

There is no doubt that an engineer will prefer the transducer designed in the first way (it is very likely that for a given length that transducer produces the purest mode), but it is worthwhile to note that, depending on the choice of ψ_m, an infinity of solutions exists.

Our second example is a mode transducer from the H_{02} mode of a rectangular waveguide into the H_{01} mode of a circular waveguide. Mode transducers between these two modes were designed a long time ago.[7] We wish to suggest an alternative solution.

ψ_m is chosen in the form of (3). The ratio of the diameter of the circle (denoted by d) to the width of the rectangle (denoted by a) is $d/a = 1.22$, determined from the equality of the cutoff wave numbers. The height of the rectangular waveguide is chosen to be equal to the diameter of the circular waveguide. Thus, the eigenfunction of the mth transmission line may be expressed as follows:

$$\psi_m(x, y, z) = g_1(z) \cos \frac{2\pi}{a} y + g_2(z) J_0 \left(\frac{2\pi}{a} \sqrt{x^2 + y^2} \right) \quad (8)$$

where x, y, z are Cartesian coordinates.

In this case, the intermediate cross sections of the transducer cannot be determined by simple considerations. Either a numerical solution of the differential equation (2) is necessary, or the following graphical method can be used.

The ψ_m = constant curves representing the lines of electric intensity are plotted in Fig. 3 for given values of $g_1(z)$ and $g_2(z)$. A possible boundary intersects perpendicularly these electric lines. A further consideration is that the resulting surface between the prescribed cross sections should be a smooth one. Taking account of these requirements, Fig. 4 shows four cross sections (initial, two intermediate, and final) of the transducer designed. One can see from the changing picture of the electric lines how the H_{02} mode of the rectangular waveguide is transformed into the H_{01} mode of the circular waveguide.[8]

Let us choose for the third example a mode transducer, which—although it has no practical importance at the moment—illustrates how powerful the method is.

[7] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," McGraw-Hill Book Co., Inc., New York, N. Y., p. 340; 1948.

[8] A transducer of this type has been constructed and tested by Dr. Y. Klinger. It is 3 inches long, $\frac{5}{8}$ inch diameter at the circular end. Preliminary measurements at the wavelength 9 mm indicate a power level in the desired H_{01} circular mode of about 95 per cent.

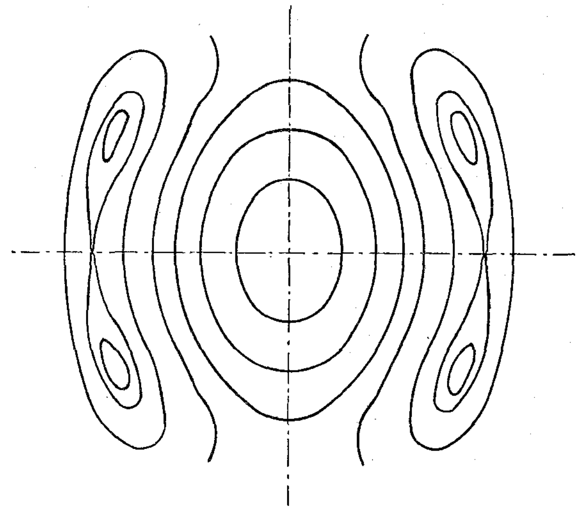

Fig. 3—The ψ_m = constant curves, $g_1(z) = 0.246$, $g_2(z) = 0.8$.

The cross sections of the two uniform waveguides (rectangle and isosceles right-angled triangle) to be connected and the electric lines of the desired modes may be seen in Figs. 5(a) and 5(e). The mode in the triangular waveguide is the same as an H_{11} mode in a square waveguide.

In this case— in our opinion—intuition fails and the cross sections of the transducer cannot be guessed, while the application of the proposed design method leads to a direct result.

The height of the rectangle (denoted by b) is chosen to be equal to the sides of the triangle. The equality of the eigenvalues is assumed by the choice $w = b/\sqrt{2}$, where w is the width of the rectangle. Constructing the eigenfunction of the mth transmission line in the same way as before, we obtain

$$\psi_m(x, y, z) = g_1(z) \cos \frac{\pi y}{w} + g_2(z) \cos \frac{\pi x}{w\sqrt{2}} \cos \frac{\pi y}{w\sqrt{2}} \cdot \quad (9)$$

Three intermediate cross sections of the transducer may be seen in Figs. 5(b), (c), and (d). Having seen these figures, one can imagine how the transducer works; *i.e.*, the application of the method helps to build up a deeper physical insight.

IV. Conclusion

It has been shown that for a gradual transducer the outgoing mode t_m^B can be determined from the incident mode t_m^A (exceptions are treated in the Appendix). The gradually changing field configuration in the transducer is represented by an appropriately chosen function, the eigenfunction of the mth transmission line. When this theorem is inverted, the eigenfunction of the mth transmission line is determined from the incident and outgoing modes, and with its aid the surface of the transducer is also determined.

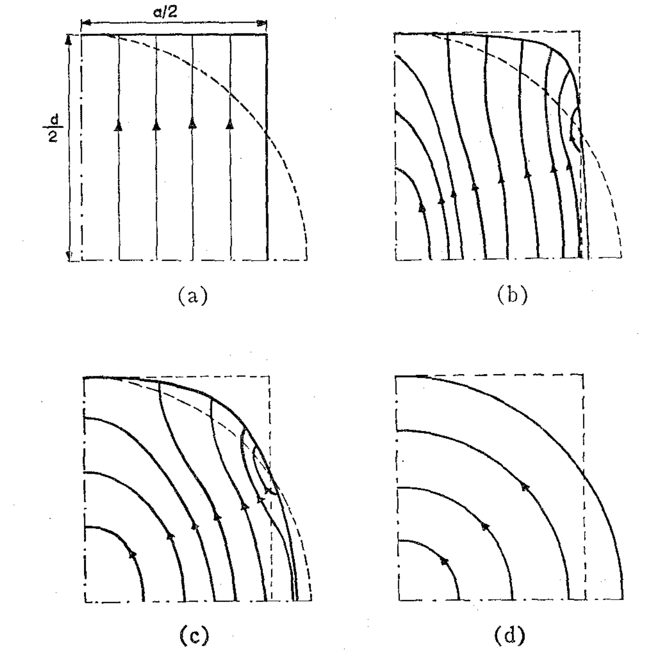

Fig. 4—Four cross-sections of a transducer from the $H_{02}^{\square}$ mode to the H_{01}° mode. (a) $g_1(z)=1$, $g_2(z)=0$; (b) $g_1(z)=0.739$, $g_2(z)=0.4$; (c) $g_1(z)=0.246$, $g_2(z)=0.8$; (d) $g_1(z)=0$, $g_2(z)=1$.

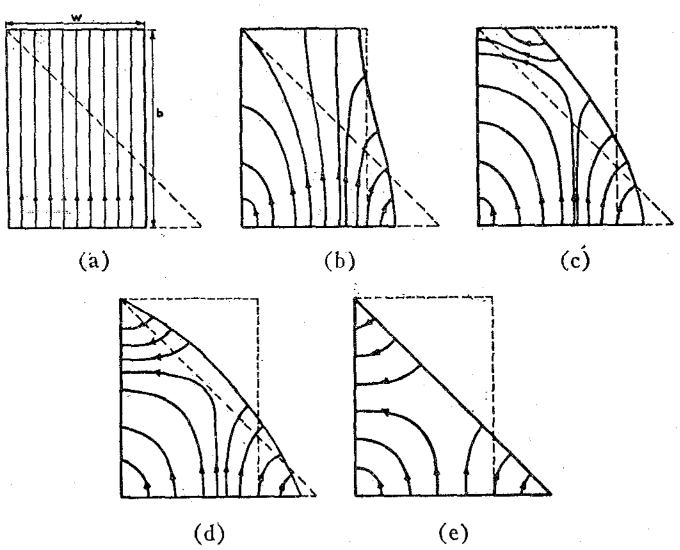

Fig. 5—Five cross-sections of a transducer from the $H_{01}^{\square}$ mode to the $H_{11}^{\triangle}$ mode. (a) $g_1(z)=1$, $g_2(z)=0$; (b) $g_1(z)=0.6$, $g_2(z)=0.4$; (c) $g_1(z)=0.37$, $g_2(z)=0.63$; (d) $g_1(z)=0.2$, $g_2(z)=0.8$; (e) $g_1(z)=0$, $g_2(z)=1$.

Three examples have been worked out to illustrate the design procedure. The electric lines in intermediate cross sections are plotted; these show how the transducer works.

The paper presents a systematic approach to the design of mode transducers, but certainly leaves many questions unanswered, a few of which are listed here.

1) In choosing a ψ_m, does a transducer exist between the required cross sections of the uniform waveguides for which ψ_m is the eigenfunction of the mth transmission line?
2) Which is the best choice of ψ_m?
3) When choosing ψ_m in the form (3), what is the best choice of $g_1(z)$ and $g_2(z)$?

The answers to these questions do not seem to be simple ones, but there is no reason to suppose that the optimum design (in one or other sense) of mode transducers is prohibitive.

Appendix

There are two types of cases when the conclusions of Section II are not valid. Both are consequences of degeneracy.

1) If a mode at a certain cross section of the transducer can be represented as the superposition of two modes which have the same cutoff wave numbers, then, because of a change in the boundary, the two components might separate (in the equivalent circuit this means that a transmission line is split into two). This may happen, for example, with a (not circularly symmetrical) mode in a circular pipe. If the circular waveguide is deformed into an elliptical waveguide, and the deformation does not take place along one of the axes of symmetry, then two separate modes with different velocities will propagate in the elliptical waveguide.[9]

2) If in the equivalent circuit of the transducer there exists another transmission line whose cutoff wave number agrees with that of the mth transmission line for every value of z, and these two transmission lines are coupled, then this coupling can never be neglected. The power is fluctuating between these two transmission lines.[10] This happens, for example, in a bent circular waveguide, where the H_{01} and E_{11} modes (each representing a transmission line) have the same cutoff wave number and are coupled through the boundary.

It is unlikely that in the design of a mode transducer either of these cases will arise. Nevertheless, it must be borne in mind that these effects can cause the failure of the method.

Acknowledgment

The authors wish to thank L. Lewin, Dr. Y. Klinger, and Dr. A. E. Karbowiak for many interesting discussions. Acknowledgment is also made to Standard Telecommunication Laboratories for permission to publish the paper.

[9] L. J. Chu, "Electromagnetic waves in elliptic hollow pipes of metal," *J. Appl. Phys.*, vol. 9; September, 1938.

[10] S. E. Miller, "Coupled wave theory and waveguide applications," *Bell Sys. Tech. J.*, vol. 33, pp. 661–719; May, 1954.